ON THE ENERGY OF ACOUSTIC WAVES PROPAGATING IN MOVING MEDIA

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The problem of the damping of shock-waves at great distances from their point of origin was first examined by Crussard, who found the asymptotic law for their motion in a straight channel of constant-cross-sectional area [1]. The laws of propagation of weak cylindrical and spherical shock-waves in a homogeneous medium were established by different methods in the works of Landau [2], Khristianovich [3], Sedov [4], Whitham [5], and a number of other authors. The first solution of the problem of the behavior of a shock-wave of small amplitude in a nonhomogeneous medium is given in the paper of Whitham [6], where it is assumed that the distribution of the parameters of the gas satisfies spherical symmetry both in the equilibrium state and in that arising from the motion of the wave. Otterman [7] investigated the damping of a shock-front in a stratified quiescent medium of constant temperature. The general problem of the propagation of shock-waves in a moving nonhomogeneous medium was treated in the works of Gubkin [8], Polianskii [9], and in [10]; there no restrictions were imposed on the character of the initial distribution of pressure, density, temperature, or speed of the gas particles.

Asymptotic laws for the variation of gas parameters along a shock-wave at great distances from a body in a steady uniform supersonic stream have been given by Landau [2] and Whitham [11,12] under the assumption that the flow is either plane or axisymmetric. The interaction of shock waves in a plane parallel stream with a stratified structure was investigated by Riley [13]. The behavior of shock-waves far from bodies in arbitrary supersonic flows was studied in [14], where the difference between unsteady and steady problems was also discussed. In the paper of Friedrichs [15] a theory of the second approximation was developed, which permits establishing with a very high degree of accuracy the laws of damping of shock-waves in flows whose parameters are determined by two independent variables. In the parts relating to unsteady motion, the results of Friedrichs agree with those of Crussard [1].

It must be remarked that all the cited works are based in an essential way upon the assumption that the entropy of a particle is unchanged in transition through the shock-front. This assumption is justified, because the change in entropy behind a shock-wave of small amplitude is proportional to the cube of the change of any of the other quantities determining the state of the gas, and the investigation is subject to equations retaining only leading terms. In the monograph of Zel'dovich [16] it is shown for the example of one-dimensional motion taking place in a cylindrical channel that the problem of damping of a shock-wave can be treated by some method based on calculation of the dissipation of energy at the front. The value of the dissipation is completely determined by the change in energy, which it was shown could be neglected in all other methods. An analogous idea was expressed by DuMond, Cohen, Panofsky, and Deeds [17]. In [18] the indicated method was applied to the solution of the problem of motion of a shock-wave of small amplitude in a nonhomogeneous quiescent medium. It was shown that in this case also the difference between the law of dissipation of a weak shock-wave and the acoustic one is completely determined by the quantity of energy irreversibly transformed into heat. Calculation of the dissipation of energy at shock-waves is also the basis of the work of Lighthill [19] and Phythian [20], in which the accuracy of Friedrichs' theory [15] is investigated.

In the present work, which is a development of [18], an equation is derived that is a consequence of the law of conservation of energy applied to acoustic waves propagating in an arbitrary nonhomogeneous moving medium. All further investigations are carried out on the basis of this equation, together with the additional assumption that the width of the region of disturbed gas motion is small compared with the radius of curvature of the shock-front and with the distance at which the parameters of the initial medium are essentially changed. It permits significant simplification of the equation mentioned above and, by integration, leads to a formula expressing the change in pressure increment at a shock-front in the approximation of geometrical acoustics [21-23]. In addition, by comparison with acoustics the dissipation of the amplitude of the wave is accounted for, as before, by the dissipation of energy arising at the shock transition. Transforming the line of reasoning, it becomes possible to reduce the problem of propagation of waves of small amplitude in an arbitrary medium to the solution of a system of two ordinary differential equations for the magnitude of the pressure increment behind the front and a measure of the acoustic impulse of the wave.

1. Acoustic approximation. The change in energy of a gas moving in a gravitational field is determined by the equation

$$\frac{\partial}{\partial t}\rho\left(\frac{v^2}{2}+\varepsilon\right)+\operatorname{div}\rho\left(\frac{v^2}{2}+w\right)\mathbf{v}-\rho\mathbf{v}\mathbf{g}=0 \tag{1.1}$$

Here t is the time, $\mathbf{v}$ the velocity of a particle, $\mathbf{g}$ the acceleration of gravity, ρ the density, ε the specific internal energy, and w the specific enthalpy. The value of w is related to ε and the pressure p by the formula

$$\rho w = \rho\varepsilon + p$$

We consider the propagation of an acoustic wave in a nonhomogeneous moving medium. We will suppose that in the undisturbed state the velocity $\mathbf{v}_0$ of a particle, the pressure p_0, density ρ_0, specific entropy s_0, internal energy ε_0, and specific enthalpy w_0 do not change with time and are given as functions of the coordinates x_i. In this case the equations of gas dynamics, written for the initial state of the gas, take the form

$$\operatorname{div}\rho_0\mathbf{v}_0=0, \qquad (\mathbf{v}_0\nabla)\,\mathbf{v}_0+\rho_0^{-1}\operatorname{grad}p_0=\mathbf{g}, \qquad \mathbf{v}_0\operatorname{grad}s_0=0 \tag{1.2}$$

We simplify Equations (1.1) using the smallness of the amplitude of an acoustic wave. For this purpose we expand the quantity ρw in a series, where as independent thermodynamic variables we choose the pressure p and specific entropy s. Let T denote the temperature. According to a known thermodynamic relation

$$dw = T\,ds + \rho^{-1}\,dp$$

Using this equality, we write the desired expansion for ρw, in which we retain terms of first and second orders of smallness:

$$\begin{aligned}\rho w = {} & \rho_0 w_0 + \left(1+\frac{w_0}{a_0^2}\right)p' + \left[\rho_0 T_0 + w_0\left(\frac{\partial\rho}{\partial s_0}\right)_p\right]s' + \\ & + \frac{1}{2}\left[\frac{1}{\rho_0 a_0^2} + w_0\left(\frac{\partial^2\rho}{\partial p_0^2}\right)_s\right]p'^2 + \left(\frac{T_0}{a_0^2} + w_0\frac{\partial^2\rho}{\partial p_0\partial s_0}\right)p's' + \\ & + \frac{1}{2}\left[2T_0\left(\frac{\partial\rho}{\partial s_0}\right)_p + \frac{\rho_0 T_0}{c_{p0}} + w_0\left(\frac{\partial^2\rho}{\partial s_0^2}\right)_p\right]s'^2\end{aligned}$$

Here a is the speed of sound, c_p the specific heat at constant pressure, p' and s' the deviations of pressure and entropy from their initial values, and the subscript zero denotes values in the undisturbed state.

We substitute the given expression into Equation (1.1), which expresses the law of conservation of energy in a moving continuous medium. To simplify the resulting relation we use Formula (1.2), and also the continuity equation, Euler's equations, and the equation of conservation of the entropy of a particle, in which it is necessary to retain quadratic terms. As a result we have

$$\frac{\partial}{\partial t}\frac{1}{2}\left(\rho_0 v'^2 + \frac{p'^2}{\rho_0 a_0^2}\right) + \operatorname{div}\left[p'\mathbf{v}' + \frac{1}{2}\left(\rho_0 v'^2 + \frac{p'^2}{\rho_0 a_0^2}\right)\mathbf{v}_0\right] +$$
$$+ \rho_0 \mathbf{v}' (\mathbf{v}' \ \nabla) \mathbf{v}_0 - \frac{m_0 - 1}{\rho_0^2 a_0^4} p/2\, \mathbf{v}_0 \operatorname{grad} p_0 -$$
$$- p's'\mathbf{v}_0 \cdot \operatorname{grad}\left[\frac{1}{\rho_0 a_0^2}\left(\frac{\partial p}{\partial s_0}\right)_\rho\right] + \frac{1}{\rho_0 a_0^2}\left(\frac{\partial p}{\partial s_0}\right)_\rho s'\mathbf{v}' \cdot \operatorname{grad} p_0 = 0 \qquad (1.3)$$

Here $\mathbf{v}'$ is the perturbation velocity vector of a particle of the medium, and the dimensionless coefficient m_0 is given by the formula

$$m_0 = \frac{1}{2\rho_0^3 a_0^2}\left(\frac{\partial^2 p}{\partial V_0^2}\right)_s$$

where V denotes the specific volume, equal to $1/\rho$. For a perfect gas with c_p = const and γ denoting Poisson's adiabatic exponent, the coefficient m_0 is equal to $(\gamma + 1)/2$, and the quantity

$$\frac{1}{\rho_0 a_0^2}\left(\frac{\partial p}{\partial s_0}\right)_\rho = \frac{1}{c_p}$$

If the medium is quiescent in its undisturbed state ($\mathbf{v}_0 = 0$), we obtain the equations derived in [18]; if moreover the characteristics of the gas in the equilibrium state are constant in all directions then the equations that follow from (1.3) express, as is known [22,23], the law of conservation of energy for an acoustic wave propagating in a uniform medium.

We now consider the motion of a thin acoustic wave, that is, a wave in which the width of the zone of disturbance of the flow λ_* is small compared with the principal radii of curvature of the shock-front and with the distance at which the parameters of the initial medium are significantly changed. The amplitude and direction of such a wave scarcely deviate from their earlier values in a distance of the order of the width λ_* of the disturbed region. As $\lambda_* \to 0$, relations hold to a first approximation among the parameters of the gas that are determined by plane flow of impulse of small amplitude:

$$\mathbf{v}' = v'\mathbf{n}, \qquad v' = \frac{p'}{\rho_0 a_0}, \qquad s' = 0 \qquad (1.4)$$

Here $\mathbf{n}$ is the unit vector normal to the wave front. Equation (1.4) is valid also for a weak shock transition.

In a thin acoustic wave the density of energy e and the density of acoustic energy flux $\mathbf{q}$ should also be connected by a relation that characterizes one-dimensional flow of impulse:

$$\mathbf{q} = e\,(a_0\mathbf{n} + \mathbf{v}_0) \tag{1.5}$$

Substituting Equation (1.4) into the relation (1.3) we obtain the basic equation of geometrical acoustics

$$\frac{\partial}{\partial t}\frac{p'^2}{\rho_0 a_0^2} + \operatorname{div}\frac{p'^2}{\rho_0 a_0^2}(a_0\mathbf{n} + \mathbf{v}_0) + \frac{p'^2}{\rho_0 a_0^2}\left[\mathbf{n}\,(\mathbf{n}\nabla)\,\mathbf{v}_0 + (m_0 - 1)\operatorname{div}\mathbf{v}_0\right] = 0 \tag{1.6}$$

We introduce the derivative along a ray on which an element of the wave moves

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (a_0\mathbf{n} + \mathbf{v}_0)\nabla$$

The coordinates x_i of the ray and the variation along it of the components n_i of the normal $\mathbf{n}$ are determined by the solution of a system of ordinary differential equations

$$\frac{dx_i}{dt} = a_0 n_i + v_{0i}, \qquad \frac{dn_i}{dt} = (n_i n_j - \delta_{ij})\left(\frac{\partial a_0}{\partial x_j} + n_k\frac{\partial v_{0k}}{\partial x_j}\right) \tag{1.7}$$

$$(\delta_{ij} = 0 \text{ for } i \neq j \text{ and } \delta_{ij} = 1 \text{ for } i = j)$$

Here we use the customary tensor notation of summation over the repeated indices j and k, which take the values 1, 2, 3.

Writing the second of Equations (1.7) in vector form and taking its scalar product with $\mathbf{v}_0$ we obtain

$$\mathbf{v}_0\frac{d\mathbf{n}}{dt} = [(\mathbf{v}_0\mathbf{n})\,\mathbf{n} - \mathbf{v}_0]\quad \operatorname{grad} a_0 + (\mathbf{v}_0\mathbf{n})\,[\mathbf{n}\,(\mathbf{n}\nabla)\,\mathbf{v}_0] - \mathbf{n}\,(\mathbf{v}_0\nabla)\,\mathbf{v}_0]$$

We take advantage of this relation and rewrite the coefficient of the free term

$$\mathbf{n}\,(\mathbf{n}\nabla)\,\mathbf{v}_0 + (m_0 - 1)\operatorname{div}\mathbf{v}_0 = (a_0\mathbf{n} + \mathbf{v}_0)\operatorname{grad}\ \ln\frac{a_0 + \mathbf{v}_0\mathbf{n}}{a_0}$$

Now the basic equation of geometric acoustics takes the form

$$\frac{\partial e}{\partial t} + \operatorname{div}(a_0\mathbf{n} + \mathbf{v}_0)\,e = 0, \qquad e = (a_0 + \mathbf{v}_0\mathbf{n})\frac{p'^2}{\rho_0 a_0^3} \tag{1.8}$$

Equation (1.8) expresses the law of conservation of energy for propagation of a thin wave of small amplitude in a moving medium. It shows that the relationship (1.5) between the quantities e and $\mathbf{q}$ actually holds. This circumstance is explained by the fact that upon passage of a short acoustic impulse through a given point of space the energy flux through unit area inclined at this point normal to the direction of the ray speed $a_0\mathbf{n} + \mathbf{v}_0$ is completely determined by the transferred quantity together with the wave energy. We note that the value of energy density e of an acoustic wave propagating in a moving medium differs by the factor $(a_0 + \mathbf{v}_0\mathbf{n})/a_0$ from the value of the energy density of a wave passing through a motionless homogeneous [22,23] or nonhomogeneous [18] medium. When $\mathbf{v}_0 = 0$ then

$$(a_0 + \mathbf{v}_0\mathbf{n}) \,/\, a_0 = 1.$$

An equation analogous to (1.8) was found by Blokhintsev [23] in the study of acoustic oscillations whose parameters are given in the product form $A \exp(i\psi)$. Here A is a slowly varying function of the coordinates and time, and the phase $\psi(t, x_i)$ of the wave is an "almost linear" function.

Integration of Equation (1.8) gives

$$e = e_0 \exp\Big[-\int_{t_0}^{t} \operatorname{div}\,(a_0\mathbf{n} + \mathbf{v}_0)\, dt\Big] \tag{1.9}$$

Here e_0 denotes the density of acoustic energy at the initial instant $t = t_0$.

We consider an elementary ray tube, that is a tube of small cross-sectional area whose generators are rays. In the case when $\mathbf{v}_0 \neq 0$, the direction of the normal $\mathbf{n}$ to the wave front does not coincide with the ray direction. We denote by $u_0 = \sqrt{[(a_0\mathbf{n} + \mathbf{v}_0)^2]}$ the ray speed with which the wave travels through space. Its projection u_{0n} onto the normal $\mathbf{n}$ is $u_{0n} = a_0 + \mathbf{v}_0\mathbf{n}$. The area f of an element of the front of acoustic impulse contained within the ray tube is related to the area σ of its cross-section by the equation

$$f = \frac{u_0}{u_{0n}}\sigma \tag{1.10}$$

Using the last relation, it is easy to show that

$$\exp\Big[\int_{t_0}^{t} \operatorname{div}\,(a_0\mathbf{n} + \mathbf{v}_0)\, dt\Big] = \frac{u_{0n}}{u_{00n}}\frac{f}{f_0}$$

Here u_{00n} and f_0 denote respectively the projection of the ray speed onto the normal and the area of the front of elementary impulse under consideration at the initial instant $t = t_0$. In the case when $\mathbf{v}_0 = 0$ in the whole space, we then have [21]

$$f = f_0 \exp\Big(\int_{t_0}^{t} a_0 \operatorname{div}\mathbf{n}\, dt\Big)$$

Equation (1.9) now takes the form

$$e = e_0 \frac{u_{00n} f_0}{u_{0n} f} \tag{1.11}$$

Equation (1.11) is valid for the entire field of disturbed flow, including also the acoustic approximation for a shock-front being considered. It determines the variation of acoustic intensity along the path of an element of the wave. Let p_0', a_{00} and ρ_{00} denote respectively the pressure increment, equilibrium speed of sound, and equilibrium density at the initial instant $t = t_0$. From Equation (1.11) follows the law of variation of the amplitude of an acoustic wave

$$p' = p_0' \frac{a_0 u_{00n}}{a_{00} u_{0n}} \sqrt{\frac{\rho_0 a_0 f_0}{\rho_{00} a_{00} f}} \tag{1.12}$$

If Equation (1.6) is integrated along a ray, the expression for the quantity p' can be put into the following form (which was obtained from different considerations by Keller [21])

$$p' = p_0' \sqrt{\frac{\rho_0 a_0}{\rho_{00} a_{00}}} \frac{1}{L}, \qquad L = \exp\left\{\frac{1}{2}\int_{t_0}^{t} [a_0 \operatorname{div} \mathbf{n} + k_0 \operatorname{div} \mathbf{v}_0 + \mathbf{n}(\mathbf{n}\nabla)\mathbf{v}_0] dt\right\}$$

$$(k_0 = 2m_0 - 1) \tag{1.13}$$

Comparison of Equations (1.12) and (1.13) gives

$$L = \frac{a_{00} u_{0n}}{a_0 u_{00n}} \sqrt{\frac{f}{f_0}}$$

We give another derivation of Equation (1.12). For this we consider the motion of an acoustic impulse along an elementary ray tube, where it is assumed that the length of the tube is sufficiently great that in the time of motion the wave does not intersect its end sections σ_1 and σ_2. We suppose that all incremental quantities in the acoustic impulse of the shock-front have a triangular profile. Although this approximation is not essential, it allows the following calculations to be simplified.

We integrate Equation (1.8) over a fixed volume V_0 bounded by the surface of the ray tube under consideration and its two end-sections σ_1 and σ_2. Since the volume V_0 does not change with time, we obtain

$$\frac{\partial}{\partial t} \int_{V_0} e dV = 0$$

Using Equation (1.10) and the assumption made above concerning the character of the distribution of the parameters of the gas in the zone of disturbed motion, we have

$$\frac{d}{dt}\frac{\lambda f u_{0n} p'^2}{\rho_0 a_0^3} = 0 \qquad (1.14)$$

Here λ denotes the wave lenght in the approximation of geometrical acoustics, and p' the pressure increment at the shock-front.

The quantity

$$E = \frac{1}{3}\frac{\lambda f u_{0n} p'^2}{\rho_0 a_0^3}$$

is the total energy of an elementary acoustic impulse contained within the ray tube of cross-sectional area σ. The equation written above therefore shows that the total energy of each elementary impulse remains constant in the present approximation. An analogous derivation is given in [18] for waves propagating in a quiescent medium.

We note that the wave length λ in the acoustic approximation is equal to [9]

$$\lambda = \lambda_0 \frac{u_{0n}}{u_{00n}} \qquad (1.15)$$

Integration of Equation (1.14) using the last relation leads to Equation (1.12).

We consider now the motion of a shock-wave of small amplitude in the approximation beyond that of geometric acoustics. In this approximation the speed of a shock-front is different from the speed of propagation of an acoustic wave, and its amplitude is damped according to a different law than (1.12) or (1.13). Zel'dovich observed [16], that the damping of a plane shock-wave in a straight tube is entirely determined by the dissipation of energy arising from wave compression. In [18] it was shown that the law of damping of a shock-wave of arbitrary form that propagates in a nonhomogeneous quiescent medium is also directly related to the quantity of acoustic energy irreversibly transformed into heat. We show that this fact is general; that is, the nonlinear laws of damping of shock-wave in arbitrarily moving media are explained by the dissipation of energy arising at shock transitions.

2. Dissipation of energy at a shock-front. To simplify the calculations we will assume as before that the perturbation quantities in the acoustic impulse bounded by the shock-front have triangular profiles. Let p_*' denote the amplitude of the shock-wave and λ_* the length of the impulse in the approximation beyond geometrical acoustics. Using the results obtained, we can show that the amplitude p_*' of the shock-wave is determined by the Formula (8) to (10).

$$p'_* = p'_0 \sqrt{\frac{\rho_0 a_0^3 f_0}{\rho_{00} a_{00}^3 f}} \frac{u_{00n}}{u_{0n}} \left(1 + \frac{p'_0 u_{00n}^2 \sqrt{f_0}}{\lambda_0 \sqrt{\rho_{00} a_{00}^3}} \int_{l_0}^{l} \frac{m_0 \sqrt{a_0} dl}{u_0 u_{0n}^2 \sqrt{\rho_0 f}}\right)^{-1/2} \tag{2.1}$$

Here dl is the element of length of the ray, equal to $u_0 dt$. The wave length λ_* varies according to the Equation (9) to (10)

$$\lambda_* = \lambda_0 \frac{u_{0n}}{u_{00n}} \left(1 + \frac{p'_0 u_{00n}^2 \sqrt{f_0}}{\lambda_0 \sqrt{\rho_{00} a_{00}^3}} \int_{l_0}^{l} \frac{m_0 \sqrt{a_0} dl}{u_0 u_{0n}^2 \sqrt{\rho_0 f}}\right)^{1/2} \tag{2.2}$$

We calculate the change in unit time of the total energy of an elementary acoustic impulse contained within the ray tube of cross-sectional area σ. The area f of the shock-front bounded by this tube is related to σ by Equation (1.10), so that the total energy E_* of the acoustic impulse with triangular profile for the pressure increment is, according to the results of the preceding section, equal to

$$E_* = \frac{1}{3} \frac{\lambda_* f u_{0n} p'^2_*}{\rho_0 a_0^3}$$

According to relations (2.1) and (2.2) this expression can be transformed into the form

$$E_* = \frac{1}{3} \frac{p'^2_0 \lambda_0 f_0 u_{00n}}{\rho_{00} a_{00}^3} \left(1 + \frac{p_0' u_{00n}^2 \sqrt{f_0}}{\lambda_0 \sqrt{\rho_{00} a_{00}^3}} \int_{l_0}^{l} \frac{m_0 \sqrt{a_0} dl}{u_0 u_{0n}^2 \sqrt{\rho_0 f}}\right)^{-1/2}$$

The change in energy of the impulse under consideration is determined by the derivative dE_*/dt, whose value is easily calculated using Equation (2.1). We have finally

$$\frac{dE_*}{dt} = -\frac{1}{6} \frac{m_0 u_{0n}}{\rho_0^2 a_0^4} f p'^3_* \tag{2.3}$$

We show that this change in energy of an elementary acoustic impulse is proportional to the magnitude of its dissipation at the shock compression. The value of this dissipation, produced by viscosity and heat conduction, may be calculated within the framework of an ideal fluid using the entropy change s_*' that occurs at the shock-front. The value of s_*' is a quantity of third order of smallness with respect to the pressure increment p_*', given by the equation

$$s'_* = \frac{1}{6} \frac{m_0}{\rho_0^3 a_0^4 T_0} p'^3_*$$

Using this relation we find the energy that is dissipated in the form of heat in unit time at an element of the shock-front of area f. Let Q_* denote the dissipated energy. Its change in unit time is evidently given by the equation

$$\frac{dQ_*}{dt} = \frac{1}{6}\frac{m_0}{\rho_0^2 a_0^3} f p'^3_* \tag{2.4}$$

which agrees, except for a factor u_{0n}/a_0 taken with reversed sign, with the expression appearing on the right-hand side of Equation (2.3). As was already observed above, the value of the energy density e of an acoustic wave differs by this factor when propagating in a quiescent and a moving medium; at the same time Equation (2.4) for dQ_*/dt always maintains its same form. In fact, in the expression for the flux of mass across a shock-front there appears the normal component of the velocity vector of particles in a system of coordinates moving with the discontinuity surface.

For weak shock-waves this velocity component agrees to a first approximation with the speed of sound a_0. When $\mathbf{v}_0 = 0$, then

$$dE_* / dt = - dQ_* / dt$$

We now invert our reasoning so that, avoiding the calculation of the pressure increment p_*' in the approximation of geometric acoustics, we can immediately find the law of damping of shock-waves of small amplitude moving in nonhomogeneous media. The change in energy of an elementary acoustic impulse of length λ_* contained within a ray tube of cross-sectional area σ and having at the front a pressure increment p_*' is determined on the basis of the equations given above as

$$\frac{d}{dt}\left(\frac{\lambda_* u_{0n} f}{\rho_0 a_0^3} p'^2_*\right) = -\frac{1}{2}\frac{m_0 u_{0n}}{\rho_0^2 a_0^4} f p'^3_* \tag{2.5}$$

The quantity λ_* appearing here satisfies the relation [10]

$$\frac{d\lambda_*}{dt} = \frac{\lambda_*}{u_{0n}}\frac{du_{0n}}{dt} + \frac{1}{2}\frac{m_0}{\rho_0 a_0} p'_* \tag{2.6}$$

The system of ordinary differential equations (2.5) and (2.6) should be integrated with the initial conditions

$$p'_* = p'_0, \qquad \lambda_* = \lambda_0 \quad \text{at} \quad t = t_0 \tag{2.7}$$

The solution of the system of Equations (2.5) and (2.6) satisfying the conditions (2.7) can be obtained in an elementary way. We divide the first of these equations by the quantity $\lambda_* f u_{0n} p_*'^2/(\rho_0 a_0^3)$ and the second by λ_*. Combining the resulting expressions we find

$$\frac{d}{dt}\ln\frac{f u_{0n}\lambda_*^2 p'^2_*}{\rho_0 a_0^3} = \frac{d}{dt}\ln u_{0n}$$

Integrating the previous equation, taking account of the initial conditions, gives

$$\lambda_* p'_* = \lambda_0 p'_0 \sqrt{\frac{\rho_0 a_0^3 f_0}{\rho_{00} a_{00}^3 f}} \tag{2.8}$$

This relation expresses the law of variation of the impulse $J_* = 1/2\ \lambda_* f p_*'/u_{0n}$ of an element of positive phase of the wave in its motion along the ray tube.

We now multiply Equation (2.6) by λ_*, replacing the product $\lambda_* p_*'$ by Expression (2.8). As a result we obtain a linear equation in λ_*^2

$$\frac{d\lambda_*^2}{dt} = \frac{2\lambda_*^2}{u_{0n}} \frac{du_{0n}}{dt} + \frac{m_0 \lambda_0 p'_0 \sqrt{a_0 f_0}}{\sqrt{\rho_{00} a_{00}^3 \rho_0 f}}$$

Using the initial conditions, we obtain as a solution of the preceding equation Formula (2.2), determining the wave length λ_*. The pressure increment p_*' at the shock-front is found, with the use of (2.8), in the form (2.1).

3. Steady flows. As was shown in [14], the results obtained above for unsteady shock-waves propagating in a nonhomogeneous medium cannot be applied immediately to the calculation of steady supersonic gas motion. For steady supersonic flow, which by definition is characterized by the condition $\partial/\partial t = 0$, we have

$$a_0 + \mathbf{v}_0 \mathbf{n} = 0 \tag{3.1}$$

In this case the rays lie wholly on the characteristic surface $\varphi(x_i) = 0$, whereas for unsteady processes of expansion waves the rays intersect their front $\varphi(t, x_i) = 0$ in the x_i space, never being tangent to these surfaces.

We denote by $\mathbf{n}_0$ the unit vector normal to the characteristic surface carrying the zero value of pressure increment. The equation governing the change in width of the disturbed region, which by analogy with unsteady motion we will also call the wave length and denote by the letter λ, has within the framework of geometrical acoustics the form [14]

$$\frac{d\lambda}{dt} = \lambda\, (\mathbf{n}_0 \nabla)\, (a_0 + \mathbf{v}_0 \mathbf{n}) \tag{3.2}$$

The value of the ray speed u_0 for steady flow is $\sqrt{(v_0^2 - a_0^2)}$; therefore in Equation (3.2) the time t is connected with the length l of the ray by the relation $dl = \sqrt{(v_0^2 - a_0^2)}dt$. We note that the solution of this equation is not, generally speaking, given by Equation (1.15), which determines the wave length in unsteady processes. Equation (1.12) for the change of amplitude of an acoustic wave in unsteady processes likewise cannot be used immediately for the calculation of steady supersonic flows. In fact, in this case according to Equation (3.1) both the quantities u_{0n} and u_{00n} vanish, and Equation (1.12) loses its significance.

In order to obtain the law of variation of the pressure increment p' along a shock-wave far from a body in a steady supersonic stream, we turn again to the result of Equation (1.6). Using the relation (3.2), we transform it to

$$\operatorname{div}(a_0\mathbf{n} + \mathbf{v}_0)\, e = 0, \qquad e = \frac{a_{00}\lambda p'^2}{\lambda_0 \rho_0 a_0^3} \tag{3.3}$$

Here e denotes as before the density of acoustic energy. We note that to within the constant factor a_{00}/u_{00n} the energy density $u_{0n}p'^2/(\rho_0 a_0^3)$ of unsteady acoustic motion may be put in the same form with the aid of Equation (1.15). The density of energy of waves propagating in a fixed medium differs from the expression for energy density in the present case by the factor $\lambda a_{00}/\lambda_0 a_0$.

Integration of Equation (3.3) leads to Equation (1.9), from which follows the law for the variation of the amplitude p' of a wave in the acoustic approximation:

$$p' = p'_0 \frac{a_0}{a_{00}} \sqrt{\frac{\rho_0 a_0 u_{00} \lambda_0 \sigma_0}{\rho_{00} a_{00} u_0 \lambda \sigma}} \tag{3.4}$$

Here, as before, σ denotes the cross-sectional area of an elementary ray tube.

Henceforth, as in the case of unsteady wave motion, we will assume for simplicity that in the disturbed region bounded by the shock-front all perturbation quantities have triangular profiles. Henceforth we will denote by p' the pressure increment at the shock-wave. We now take, on the characteristic surface carrying the undisturbed values of the gas parameters, an arbitrary point that we use as the origin for the ray passing through it. Through the chosen point we pass a plane perpendicular to the ray direction, and in this plane we construct a small closed contour consisting of the line of its intersection with the shock-wave and the characteristic surface where $p' = 0$, and also of two neighboring normals to the indicated surface. We consider the ray tube whose generators pass through the sides of the rectangle consisting of the contour we have constructed. It is evident that the cross-section of such a tube has at any point the form of a parallelogram; its area can be identified with the quantity σ in Equation (3.4), since the width of the disturbed region is regarded as small compared with the principal radii of curvature of the shock-front and the distance at which the parameters of the basic stream are essentially changed. We now introduce on the characteristic surface carrying zero values of p' a small rectangle, one of whose sides serves as a side of the closed contour considered earlier. The rectangular cylinder with generators parallel to the normal to the characteristic surface bounds in the ray tube under consideration a certain element of the wave. This element as a whole is carried along the ray in the first approximation with speed u_0, and deforming acquires the shape of a parallelopiped. The dimensions of the parallelopiped in the ray direction change proportional to the ray speed u_0.

If as before we denote by f, the frontal area of an element of the wave, the following relation is easily seen to hold

$$\frac{u_0\sigma}{\lambda f}=\frac{u_{00}\sigma_0}{\lambda_0 f_0}$$

Using this, we can put Equation (3.4) into the form

$$p'=p'_0\frac{a_0\lambda_0}{a_{00}\lambda}\sqrt{\frac{\rho_0 a_0 f_0}{\rho_{00}a_{00}f}} \tag{3.5}$$

We note that Expression (1.12) is transformed to the same form using Equation (1.15).

We show that the form of Equation (3.5) is not dependent upon the chosen form of the wave element. In fact, we consider an element of the wave bounded by an arbitrary cylindrical surface with generators initially parallel to the normal to the characteristic surface carrying the undisturbed values of the parameters of the gas. We integrate Equation (3.3) over the volume of the element under consideration, which is continuously deformed as it moves along the ray tube. Taking into account the assumption made above regarding the character of the distribution of the gas parameters in the disturbed flow region, we have

$$\frac{d}{dt}\frac{a_{00}f\lambda^2 p'^2}{\lambda_0\rho_0 a_0^3}=0$$

Integration of this equation, which is analogous to Equation (1.14), leads to Equation (3.5).

This thus shows that also in the case of steady gas motion the ray tube is a channel for the transmission of energy in the disturbed flow zone.

The factor L appearing in Keller's equation (1.13) takes the form

$$L=\frac{a_{00}}{a_0}\sqrt{\frac{\lambda u_0\sigma}{\lambda_0 u_{00}\sigma_0}}=\frac{a_{00}\lambda}{a_0\lambda_0}\sqrt{\frac{f}{f_0}}$$

We turn now to the investigation of the law of damping of a shock-wave far from a body in steady supersonic flow in the approximation beyond geometrical acoustics.

Using the results obtained, we have [14]

$$p'_* = p'_0 \frac{a_0\lambda_0}{a_{00}\lambda} \sqrt{\frac{\rho_0 a_0 f_0}{\rho_0 a_{00} f_{00}}} \left(1 + \frac{p'_0\lambda_0 \sqrt{f_0}}{\sqrt{\rho_{00} a_{00}^3}} \int_{l_0}^{l} \frac{m_0 \sqrt{a_0}\, dl}{u_0 \lambda^2 \sqrt{\rho_0 f}}\right)^{-1/2} \qquad (3.6)$$

$$\lambda_* = \lambda \left(1 + \frac{p'_0\lambda_0 \sqrt{f_0}}{\sqrt{\rho_{00} a_{00}^3}} \int_{l_0}^{l} \frac{m_0 \sqrt{a_0}\, dl}{u_0 \lambda^2 \sqrt{\rho_0 f}}\right)^{1/2} \qquad (3.7)$$

Here, as in the preceding section, asterisks indicate values of the gas parameters in the approximation beyond acoustics, λ denotes the wave length in linear theory, and $dl = u_0 dt$.

The total energy E_* of an element of the wave with a triangular profile of pressure increment is equal to

$$E_* = \frac{1}{3} \frac{\lambda a_{00}}{\lambda_0} \frac{\lambda_* f p'^2_*}{\rho_0 a_0^3}$$

In accordance with Equations (3.6) and (3.7), we transform this relation into the form

$$E_* = \frac{1}{3} \frac{p'^2_0 \lambda_0 f_0}{\rho_{00} a_{00}^2} \left(1 + \frac{p'_0\lambda_0 \sqrt{f_0}}{\sqrt{\rho_{00} a_{00}^3}} \int_{l_0}^{l} \frac{m_0 \sqrt{a_0}\, dl}{u_0 \lambda^2 \sqrt{\rho_0 f}}\right)^{-1/2}$$

The change of energy in unit time for the element of the wave under consideration is given by the derivative

$$\frac{dE_*}{dt} = -\frac{1}{6} \frac{a_{00}\lambda m_0 f}{\lambda_0 \rho_0^2 a_0^4} p'^3_* \qquad (3.8)$$

This expression, taken with reversed sign, agrees to within a factor $\lambda_{00} a/\lambda_0 a_0$ with the value (2.4) of the derivative dQ_*/dt, which determines the dissipation of energy at a shock-front of area f. As noted above, the value of energy density differs by this factor in steady flow from the value for an acoustic wave passing through a quiescent medium. When the oncoming stream is uniform, $\lambda a_{00}/\lambda_0 a_0 = 1$.

Reversing the line of reasoning as in the previous section, we can obtain the law of damping of a shock-wave far from a body in a flow without calculating p' in the approximation of geometrical acoustics. We will not dwell upon the corresponding calculations in detail.

In conclusion we consider supersonic flow past a body in a stream with constant velocity $\mathbf{v}_0$ whose other parameters depend upon the coordinate x_3 measured along one of the axes perpendicular to $\mathbf{v}_0$. Such a case can be found, for example, when a body moves in a quiescent atmosphere parallel to the plane of the earth, and the density, pressure, and temperature of the air depend only on altitude. For simplicity all phenomena are described in a system of coordinates moving with the body, where the flow is steady; but for the calculation of the wave length λ in the approximation of geometrical acoustics it is convenient to change to non-moving coordinates connected with the earth. Then λ is given by Equation (1.15)

$$\lambda = \lambda_0 \frac{u_{0n}(x_3)}{u_{00n}} \tag{3.9}$$

and not by the more complicated solution of Equation (3.2). Of course one expression for λ can be transformed into the other, so that it is easy to convince oneself immediately that they check. In the present case Equation (3.2) takes the form

$$\frac{d\lambda}{dt} = \frac{\lambda}{a_0}(a_0\mathbf{n}\nabla)a_0$$

However, the operator $(a_0\mathbf{n}\nabla)$ is, as shown in Section 1, an ordinary derivative along a ray in a system of coordinates related to the earth.

Taking now d/dt to mean a derivative in this system of coordinates, we transform the preceding equation to

$$\frac{d\lambda}{dt} = \frac{\lambda}{a_0}\frac{da_0}{dt}$$

From this Equation (3.9) follows again.

Equation (1.15) for λ can be used whenever the change to the system of coordinates in which the body is moving does not violate the steadiness of the initial flow. However, the proof that this equation gives the solution of Equation (3.2) is very complicated in the general case. Without giving the corresponding arguments, we point out that it rests essentially on the connection with the equations of the bicharacteristics (1.7).

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